

Ocean Turbulence V: Mesoscale Modeling in Level Coordinates.**The effect of random nature of density**V.M. Canuto^{1,2} and M.S. Dubovikov¹¹NASA Goddard Institute for Space Studies

2880 Broadway, NY, NY, 10025

²Dept. Applied Physics, Columbia University, NY, NY, 10027**Abstract**

The main result of this paper is the derivation of a new expression for the tracer subgrid term in level coordinates S_ℓ to be employed in O-GCM. The novel feature is the proper account of the random nature of the density field which strongly affects the transformation from isopycnal to level coordinates of the variables of interest, velocity and tracer fields, their correlation functions and ultimately the subgrid terms. Such an effect was neglected in previous work. In deriving our result we made use of measured properties of vertical ocean turbulence (Gargett et al. 1981). The major new results are:

- 1) the new subgrid expression is different from that of the heuristic GM model,
- 2) $\mathbf{u}^{++}(\text{tracer}) = \frac{1}{2}\mathbf{u}^+(\text{thickness})$, where \mathbf{u}^{++} and \mathbf{u}^+ are the tracer and thickness bolus velocities. In previous models, $\mathbf{u}^{++} = \mathbf{u}^+$,
- 2) the subgrid for a tracer τ is not the same as that for the density ρ even when one accounts for the obvious absence of a diffusion term in the latter. The difference stems from a new treatment of the stochastic nature of the density,
- 3) the mesoscale diffusivity enters both locally and non-locally, as the integral over all z 's from the bottom of the ocean to the level z .

I. Introduction

It has long been suggested (Montgomery, 1940) that mixing in the ocean's stably stratified regions occurs preferentially along surfaces of constant densities, isopycnals, rather than along isopotential surfaces. More recently, Lozier *et al.* (1994) have shown that averaging temperature and salinity over pressure surfaces may give rise to water masses that do not exist in the real ocean. As discussed in the previous paper (IV), the general procedure is thus to construct mesoscale subgrid terms in isopycnal coordinates and then transform them to level coordinates. The problem is far from simple because of the *random nature of the relation between the density ρ and the coordinate z* . Previous work did not fully account for this feature: de Szoeke and Bennett (1992) recognize that "the equations of motion, continuity, thermodynamics and scalar conservations averaged macroscopically on isopycnals and expressed in level coordinates are not the same as the conventional equations Reynolds-averaged at fixed depth", but ultimately they argued that in practical applications such a difference is unimportant since ρ' may be considered a small parameter. We reach the opposite conclusion. We show that the first term $O(\rho')$ must also be taken into account since the neglect of this feature leads to some poorly defined effective density $\rho(z)$. For future reference, we shall call it "effective $\rho(z)$ approximation". The goal of this paper is to take proper *account of the random nature of the function $\rho(z)$* . The resulting subgrid is different from that of previous work.

The paper is organized as follows: in sec.II, we study the relations between the mean and the fluctuating components of the two fields $\rho(z)$ and $z(\rho)$ that follow from the random nature of $\rho(z)$. In sec.III, we obtain the expressions for the mean and fluctuating components for the velocity and tracer fields in level coordinates in terms of the same fields in isopycnal coordinates. In sec.IV, we consider the effect of the random nature of $\rho(z)$ on the subgrid for tracers in level coordinates and obtain a model independent expression for the tracer subgrid term in level coordinates S_ℓ in terms of two functions which must be modeled in isopycnal coordinates and which we discuss in sec.V. In sec.VI, we present the

final result for S_ℓ to be used in O-GCM. In sec. VII we present S_ℓ as the sum of diffusion+advection terms which differs considerably from previous expressions. In sec. VIII we discuss the difference between our expressions for the bolus velocities for tracer and densities in both coordinates with those of previous authors.

II. Relations between the random fields $\rho(z)$ and $z(\rho)$.

Consider the two fields $\rho(x,y,z,t)$ and $z(x,y,\rho,t)$ which we split in two parts representing the average and the fluctuating components,

$$\rho(z) = \bar{\rho}(z) + \rho'(z) \quad (1a)$$

$$z(\rho) = \bar{z}(\rho) + z'(\rho) \quad (1b)$$

For sake of simplicity, we shall omit the x,y dependence. For arbitrary z and ρ , we have the exact relations

$$\rho = \rho[z(\rho)], \quad z = z[\rho(z)] \quad (2)$$

Further, we assume that the fluctuating components $\rho'(z)$ and $z'(\rho)$ are sufficiently small to allow a power expansion. Using (3a,b) we then have (the subscripts denote partial derivatives)

$$\rho(z) = \rho(\bar{z} + z') = \rho(\bar{z}) + \rho_z(\bar{z})z' + \frac{1}{2}\rho_{zz}(\bar{z})z'^2 + \dots \quad (3a)$$

$$\rho_z(\bar{z}) \equiv \left. \frac{\partial \rho}{\partial z} \right|_{z=\bar{z}} \quad (3b)$$

Averaging, subtracting from the original equation and keeping terms up to $O(z'^2)$, we obtain:

$$\rho = \bar{\rho}(\bar{z}) + \delta\rho \quad (4)$$

where

$$\delta\rho = \overline{\rho'_z(\bar{z})z'} + \frac{1}{2}\overline{\rho_{zz}(\bar{z})z'^2} \quad (5)$$

and

$$\rho'_z(\bar{z}) + \bar{\rho}_{zz}(\bar{z})z'(\rho) + \delta\rho' = 0 \quad (6a)$$

$$\delta\rho' = \Delta[\rho'_z(\bar{z})z'] + \frac{1}{2}\bar{\rho}_{zz}(\bar{z})\Delta[z'^2] \quad (6b)$$

where we have defined

$$\Delta(xy) = xy - \overline{xy} \quad (7a)$$

and in Eqs.(4–6) we have used the shorthand notation

$$\bar{z} \equiv \bar{z}(\rho), \quad z' \equiv z'(\rho) \quad (7b)$$

An analogous procedure with the variable $z(\rho)$ leads to the following relations which correspond to Eqs.(4–5)

$$z = \bar{z}(\bar{\rho}) + \delta\bar{z} \quad (8a)$$

$$\delta\bar{z} = \overline{z'_{\rho}(\bar{\rho})\rho'} + \frac{1}{2}\bar{z}_{\rho\rho}(\bar{\rho})\overline{\rho'^2} \quad (8b)$$

The counterparts of relations (6) are:

$$z'(\bar{\rho}) + \bar{z}_{\rho}(\bar{\rho})\rho' + \delta z' = 0 \quad (9a)$$

$$\delta z' = \Delta[z'_{\rho}(\bar{\rho})\rho'] + \frac{1}{2}\bar{z}_{\rho\rho}(\bar{\rho})\Delta(\rho'^2) \quad (9b)$$

where we have used the notation:

$$\bar{\rho} \equiv \bar{\rho}(z), \quad \rho' \equiv \rho'(z) \quad (9c)$$

The "effective $\rho(z)$ approximation" used by previous authors neglects both terms $\delta\bar{\rho}$ and $\delta\bar{z}$ in (4) and (8a) which become

$$\rho = \bar{\rho}[\bar{z}(\rho)], \quad z = \bar{z}[\bar{\rho}(z)] \quad (10a)$$

Since in these relations ρ and z are arbitrary, we can choose

$$\rho = \bar{\rho}(z) \quad (10b)$$

Then, the second of (10a) implies that

$$z = \bar{z}(\rho) \quad (10c)$$

Relations (10b,c) imply that functions $\bar{z}(x)$ and $\bar{\rho}(x)$ are inverse of each other, a property that is fulfilled only within the approximation used above. Relations (10) are implicit in the work of de Szoek and Bennett (1993) and frequently used by several authors: for example, Eq.(14) of Gent et al. (1995), Eq.(4) of Treguier et al. (1997) and Eqs.(B.4), (B.8) and (B.12) of Smith (1999).

Consider the effect of retaining terms up to $O(\rho')$ and $O(z')$. Since Eqs.(4) and (8a) do

not contain such terms, relations (10) remain unchanged while Equations (6a) and (9a) become:

$$\rho'(z) + \bar{\rho}_z(z)z'(\rho) = 0 \quad (11a)$$

$$z'(\rho) + \bar{z}_\rho(\rho)\rho'(z) = 0 \quad (11b)$$

which proves that

$$\bar{\rho}_z(z)\bar{z}_\rho(\rho) = 1 \quad (11c)$$

This relation becomes an identity when one neglects the random nature of $\rho(z)$. Relation (11c) is frequently used and relations (11a,b) were employed, for example, by Treguier et al. (1997) and Rix and Willebrand (1996).

If we now account for terms $O(\rho'^2)$ and $O(z'^2)$, in lieu of relations (10a,b), Eqs.(4) and (8a) yield:

$$\rho - \bar{\rho}[\bar{z}(\rho)] = \delta\bar{\rho} \quad (12a)$$

$$z - \bar{z}[\bar{\rho}(z)] = \delta\bar{z} \quad (12b)$$

where $\delta\bar{\rho}$ and $\delta\bar{z}$ are given by expressions (5) and (8b) with (10b,c). Furthermore, using Eqs.(10) and (11), we obtain:

$$\delta\bar{\rho} = \bar{\rho}_z \delta\bar{z} \quad (12c)$$

We also write explicitly the second-order approximation in Eqs.(6a) and (9a). They are:

$$\rho'(z) + \bar{\rho}_z(z)z'(\rho) = -\delta\rho' = -\bar{\rho}_z(z)\delta z' \quad (13a)$$

where

$$\delta\rho' = \Delta[\rho'_z(z)z'(\rho)] + \frac{1}{2}\bar{\rho}_{zz}(z)\Delta[z'(\rho)^2] \quad (13b)$$

$$\delta z' = \Delta[z'_\rho(\rho)\rho'(z)] + \frac{1}{2}\bar{z}_{\rho\rho}(\rho)\Delta[\rho'(z)^2] \quad (13c)$$

where (10b,c) are assumed. Relations (12)–(13) have never been considered before and yet they are key to obtain the correct transformation from isopycnal to level coordinates.

III. Velocity and tracer fields in isopycnal and level coordinates.

Due to the random nature of the functions $\rho(z)$ and $z(\rho)$, the transformation of random fields like velocity and tracer from isopycnal to level coordinates is far from trivial.

To begin with, we split the fields into mean and fluctuating parts; in the case of a 3D velocity field \mathbf{v} , we write

$$\text{level:} \quad \mathbf{v}(z) = \overline{\mathbf{v}}(z) + \mathbf{v}'(z) \quad (14a)$$

$$\text{isopycnal:} \quad \mathbf{v}(\rho) = \tilde{\mathbf{v}}(\rho) + \mathbf{v}''(\rho) \quad (14b)$$

An overbar represents average in level coordinates, and a tilde represents average in isopycnal coordinates; similarly, the fluctuating components are denoted by a prime (') in level coordinates and by a double prime (") in isopycnal coordinates. For $\mathbf{z}(\rho)$, we use Eq.(1b) instead of (14b) since the only meaningful average is at fixed ρ and thus there is no need to distinguish between a bar and a tilde. The exact relation between the fields in the two coordinate system is written as

$$\mathbf{v}(z) = \mathbf{v}[\rho(z)] \quad (15)$$

Let us substitute in (15) the decomposition (1a) for $\rho(z)$ and let us expand the right hand side in power series in $\rho'(z)$. For the analysis that follows we retain only the zeroth and first powers in $\rho'(z)$. We have

$$\mathbf{v}(z) = \mathbf{v}[\bar{\rho}(z)] + \mathbf{v}_{\rho}[\bar{\rho}(z)]\rho'(z) \quad (16a)$$

Substituting Eq.(14b) and separating mean and fluctuating parts, we obtain

$$\overline{\mathbf{v}}(z) = \tilde{\mathbf{v}}[\bar{\rho}(z)] + \overline{\mathbf{v}_{\rho}[\bar{\rho}(z)]\rho'(z)} \quad (16b)$$

$$\mathbf{v}'(z) = \mathbf{v}''[\bar{\rho}(z)] + \tilde{\mathbf{v}}_{\rho}[\bar{\rho}(z)]\rho'(z) + \Delta\{\mathbf{v}_{\rho}''[\bar{\rho}(z)]\rho'(z)\} \quad (16c)$$

where

$$\tilde{\mathbf{v}}[\bar{\rho}(z)] \equiv \tilde{\mathbf{v}}(\rho)|_{\rho=\bar{\rho}(z)} \quad (16d)$$

$$\mathbf{v}''[\bar{\rho}(z)] \equiv \mathbf{v}''(\rho)|_{\rho=\bar{\rho}(z)} \quad (16e)$$

In analogy with (15), we have the relation

$$\mathbf{v}(\rho) = \mathbf{v}[z(\rho)] \quad (17)$$

Using (14b) and expanding in powers of $z'(\rho)$, we obtain

$$\tilde{\mathbf{v}}(\rho) = \overline{\mathbf{v}}[z(\rho)] + \overline{\mathbf{v}'_z[z(\rho)]z'(\rho)} \quad (18a)$$

$$\mathbf{v}''(\rho) = \mathbf{v}'[z(\rho)] + \overline{\mathbf{v}}_z[z(\rho)]z'(\rho) + \Delta\{\mathbf{v}'_z[z(\rho)]z'(\rho)\} \quad (18b)$$

Notice further that we limit ourselves to terms $O(\rho')$ and $O(f'^2)$, where f' is any other fluctuating variable. Thus, when we average bilinear terms in the fluctuations, as in Eqs.(16b) and (18a), the difference between averages in isopycnal and level coordinates is negligible provided $z=\bar{z}(\rho)$ or $\rho=\bar{\rho}(z)$ since it entails higher order fluctuations. Thus, Eq.(18a) can be rewritten as follows:

$$\tilde{\mathbf{v}}[\bar{\rho}(z)] = \bar{\mathbf{v}}(z) + \overline{\mathbf{v}'_z(z)z'[\bar{\rho}(z)]} \quad (18c)$$

However, even within this approximation, we must distinguish between the two fluctuations $\mathbf{v}'(z)$ and $\mathbf{v}''(\rho)$, Eqs.(16c) and (18b). Analogous relations can be obtained for the tracer fields $\tau(z)$ and $\tau(\rho)$: they can be obtained by substituting

$$\mathbf{v} \rightarrow \tau \quad (18d)$$

in Eqs.(16) and (18).

IV. Tracer field. Level Coordinates. Model Independent Result

When dealing with the equation for the mean tracer field $\bar{\tau}$ in level coordinates (∇_3 is the 3D gradient operator and $\mathbf{v}=\mathbf{u},\mathbf{w}$),

$$\partial_t \bar{\tau} + \bar{\mathbf{v}} \cdot \nabla_3 \bar{\tau} + S_\ell = 0 \quad (19a)$$

one must model the subgrid term ($\nabla \equiv \nabla_H$ is the two dimensional gradient)

$$S_\ell = \overline{\mathbf{v}' \cdot \nabla_3 \tau'} = \overline{\mathbf{u}' \cdot \nabla \tau'} + \overline{\mathbf{w}' \tau'_z} \quad (19b)$$

Due to the fact that it is physically easier to interpret and model variables in isopycnal coordinates, we shall first express S_ℓ in terms of correlation functions in such coordinates and then transform the result to level coordinates. The first step is the substitution of Eq.(16c) and the analogous relation for $\tau'(z)$ (obtained by substituting $\mathbf{v} \rightarrow \tau$ in 16c), into Eq.(19b). Since in isopycnal coordinates we only consider horizontal components of the velocity field, that is, $\mathbf{u}(\rho)$, we need to eliminate the vertical components of $\mathbf{v}(z)$. To this end, we use the continuity equation for $\mathbf{v}'(z)$ which yields

$$\mathbf{w}'(z) = \nabla \cdot \int_z^\infty \mathbf{u}'(z_*) dz_* \quad (20a)$$

Transforming the variables of integration so that $z_* = \bar{z}(x)$, we obtain

$$w'(z) = -\nabla \cdot \int_0^{\bar{\rho}(z)} \mathbf{u}'[\bar{z}(x)] \bar{z}_\rho(x) dx \quad (20b)$$

Substituting the operator $\nabla (\equiv \nabla_H)$ using the relation (∇_ρ is the gradient at fixed ρ)

$$\nabla = \nabla_\rho + (\nabla \rho) \partial_\rho \quad (21a)$$

we obtain

$$w'(z) = -\bar{\rho}_z(z)^{-1} \mathbf{u}'(z) \cdot \nabla \bar{\rho}(z) + \delta w'(z) \quad (21b)$$

where

$$\delta w'(z) = -\int_0^{\bar{\rho}(z)} \nabla_x \cdot \mathbf{u}'[\bar{z}(x)] \{\bar{\rho}_z[\bar{z}(x)]\}^{-1} dx \quad (21c)$$

so that

$$\frac{\partial}{\partial z} \delta w'(z) = -\nabla_\rho \cdot \mathbf{u}' - \mathbf{u}' \bar{\rho}_z \cdot \nabla_\rho (\bar{\rho}_z)^{-1} \quad (21d)$$

Let us notice that in Eqs.(20b) and (21c) we consider \mathbf{u}' and $\bar{\rho}_z$ to be functions of ρ whereas initially in Eqs.(14a) and (1a) they were defined as functions of z . The change of the independent variables is carried out via the substitution $z = \bar{z}(\rho)$. Analogously, even though the field \mathbf{v}'' is initially defined in (14b) as a function of ρ , in (16c) we express \mathbf{v}'' in terms of z via the substitution $\rho = \bar{\rho}(z)$. Actually, the detailed specifications shown in Eqs.(16c) and (21) are not necessary; they can be easily reconstructed since the choices of the independent variables are unique. As we discussed earlier, with accuracy up to the second order in the fluctuating fields, the average of the bilinear functions of fluctuating fields does not depend on which independent variable one chooses, ρ or z . Thus, for sake of simplicity, we shall frequently omit the arguments of the functions and use a overbar instead of a tilde. With this clarifications, we substitute Eqs.(21) into Eq.(19b) and obtain

$$S_\ell = \bar{\rho}_z \nabla_\rho \cdot (\bar{\mathbf{u}}' \bar{\tau}' / \bar{\rho}_z) + \frac{\partial}{\partial z} \bar{\delta w}' \bar{\tau}' \quad (22)$$

After substituting Eq.(21c), one observes the following:

- 1) in the first term there are only one-point correlation functions whereas in the second term there two-point correlations functions,
- 2) in the first and second term there are different integration and differentiation processes

whose relative importance can be estimated by using characteristic horizontal and vertical (more accurately, isopycnal and diapycnal) length scales L and H , as well as the characteristic diapycnal depth of the correlation, δH . The experimental results of Gargett et al. (1981) show that δH is of the order of tens of meters which is considerably smaller than $H \geq 10^3$ meters. In this way, we obtain that the ratio of the second to the first term in (22) is of order $\delta H/H$ and we can neglect it. Thus, we retain only the first term in (22),

$$S_\ell = \bar{\rho}_z \nabla_\rho \cdot (\overline{\mathbf{u}'\tau'}/\bar{\rho}_z) \quad (23)$$

Eq.(23) can also be derived in a different way. Multiply (21b) by τ' , average and neglect the term $\overline{\delta \mathbf{w}'\tau'}$ in accordance with what we have just discussed. As a result, we obtain

$$\bar{\rho}_z \overline{\mathbf{w}'\tau'} + \overline{\mathbf{u}'\tau'} \cdot \nabla \bar{\rho} = 0 \quad (24)$$

Using (24) in Eq.(19b), we derive (23). We further notice that (24) also implies that the subgrid tracer flux $\overline{\mathbf{v}'\tau'} = (\overline{\mathbf{u}'\tau'}, \overline{\mathbf{w}'\tau'})$ is directed along isopycnal surfaces. An similar conclusion for the density flux can be obtained from (21b):

$$\bar{\rho}_z \overline{\mathbf{w}'\rho'} + \overline{\mathbf{u}'\rho'} \cdot \nabla \bar{\rho} = 0 \quad (25)$$

Eqs.(24)–(25) express one of the basic facts of physical oceanography, namely that mixing in the stably stratified ocean occurs mostly along isopycnal surfaces.

Next, we use (16c) and its analog for the field τ , to express \mathbf{u}' and τ' in Eq.(23) in terms of \mathbf{u}'' and τ'' . This yields:

$$S_\ell = \bar{\rho}_z \nabla_\rho \cdot \overline{[(\mathbf{u}'' + \tilde{\mathbf{u}}_\rho \rho')(\tau'' + \tilde{\tau}_\rho \rho')]/\bar{\rho}_z} \quad (26)$$

which we expand as a power series in ρ' :

$$S_\ell = S_0 + S_1 + \dots \quad (27)$$

where:

$$S_0 = \bar{\rho}_z \nabla_\rho \cdot (\overline{\mathbf{u}''\tau''}/\bar{\rho}_z) \quad (28a)$$

To first order in $O(\rho')$, we further have:

$$S_1 = S_1^* + S_1^{**} \quad (28b)$$

$$S_1^* = \bar{\rho}_z \nabla_\rho \cdot [\overline{\bar{\tau}_\rho(\rho) \cdot \mathbf{u}''(\rho)\rho'}/\bar{\rho}_z] \quad (28c)$$

$$S_1^{**} = \bar{\rho}_z \nabla_\rho \cdot [\bar{\mathbf{u}}_\rho(\rho) \bar{\rho}' \bar{\tau}'(\rho) / \bar{\rho}_z] \quad (28d)$$

The last term is due to the correlation between the density and the tracer fields which interact indirectly through the velocity field. On that basis, we suggest that to order $O(\rho')$, this term may be neglected

$$S_1^{**} \approx 0 \quad (28e)$$

Next, we transform S_1^* . It is convenient to introduce the velocity (see Eq.10a of IV)

$$\mathbf{u}_1 \equiv -\partial_z (\bar{\mathbf{u}}'' \bar{\rho}' / \bar{\rho}_z) = (\bar{z}' \bar{\mathbf{u}}'')_z \quad (29)$$

Since at the bottom $z=-H$, \mathbf{u}'' vanishes, we have

$$\bar{\mathbf{u}}'' \bar{\rho}' / \bar{\rho}_z = -\int_{-H}^z \mathbf{u}_1(z') dz' \equiv \mathbf{I} \quad (30)$$

Using Eqs.(27), (28) and (30), S_ℓ expressed in isopycnal coordinates is:

$$S_\ell = S_0 + \bar{\rho}_z \nabla_\rho \cdot (\bar{\tau} \mathbf{I}) \quad (31)$$

One more step is needed to write (31) explicitly in terms of level coordinates. This is accomplished by the substitutions

$$\rho \rightarrow \bar{\rho}(z), \quad \partial_\rho \rightarrow \bar{\rho}_z^{-1} \partial_z, \quad \nabla_\rho \rightarrow \nabla - (\nabla \bar{\rho}) \bar{\rho}_z^{-1} \partial_z \quad (32)$$

Thus, Eq.(31) takes the form

$$S_\ell = w_1 \bar{\tau}_z + (\bar{\tau}_z / \bar{\rho}_z) \mathbf{u}_1 \cdot \nabla \bar{\rho} + S_* + S_0 \quad (33a)$$

where

$$w_1 = \nabla \cdot \mathbf{I}, \quad S_* = \mathbf{I} \cdot \partial_z [\bar{\tau}_z (\mathbf{L}_\rho - \mathbf{L}_\tau)] \quad (33c)$$

$$\mathbf{L}_\rho = -\bar{\rho}_z^{-1} \nabla \bar{\rho}, \quad \mathbf{L}_\tau = -\bar{\rho}_z^{-1} \nabla \bar{\tau} \quad (33d)$$

where $\mathbf{L}_{\rho, \tau}$ are density and tracer surfaces slopes. The variables \mathbf{u}_1 , \mathbf{I} and S_0 are given by (29), (30) and (28a). One must model two variables

$$\mathbf{u}_1, S_0 \quad (34)$$

which we consider next.

V. Interpretation of \mathbf{u}_1

Following Andrews and McIntyre (1976) and Andrews et al. (1987), the transport

equation for the mean density $\bar{\rho}$ is given by:

$$\partial_t \bar{\rho} + (\bar{\mathbf{u}} + \mathbf{u}^+) \cdot \nabla \bar{\rho} + (\bar{\mathbf{w}} + \mathbf{w}^+) \bar{\rho}_z + \Sigma_z = 0 \quad (35a)$$

where

$$\mathbf{u}^+ = -\partial_z \mathbf{A}, \quad \mathbf{w}^+ = \nabla \cdot \mathbf{A}, \quad \mathbf{A} = (\bar{\rho}_z)^{-1} \bar{\mathbf{u}}^T \bar{\rho}' \quad (35b)$$

$$\Sigma = \mathbf{A} \cdot \nabla \rho + \bar{\mathbf{w}}^T \bar{\rho}' \quad (35c)$$

Consider also the equation for the density variance in the adiabatic case,

$$\partial_t \bar{\rho}'^2 + 2\bar{\mathbf{u}}^T \bar{\rho}' \cdot \nabla \rho + 2\bar{\mathbf{w}}^T \bar{\rho}' \bar{\rho}_z + \bar{\mathbf{u}} \cdot \nabla (\bar{\rho}'^2) + \bar{\mathbf{w}} \partial_z (\bar{\rho}'^2) + \partial_z (\bar{\mathbf{w}}^T \bar{\rho}'^2) + \nabla \cdot \bar{\mathbf{u}}^T \bar{\rho}'^2 = 0 \quad (35d)$$

If we neglect terms $O(\bar{\rho}'^2)$, only the second and third term in (35d) survive, which leads to Eq.(25) which in turn implies that

$$\Sigma = 0 \quad (35e)$$

Consequently, Eq.(35a) becomes

$$\partial_t \bar{\rho} + (\bar{\mathbf{u}} + \mathbf{u}^+) \cdot \nabla \bar{\rho} + (\bar{\mathbf{w}} + \mathbf{w}^+) \bar{\rho}_z = 0 \quad (35f)$$

where \mathbf{u}^+ can now be interpreted as a *density bolus velocity in level coordinates*. We can further notice that using (29), (35b) and (18b) we obtain

$$\mathbf{u}_1 - \mathbf{u}^+ = \partial_z (\mathbf{u}_z \rho'^2 / \bar{\rho}_z^2) \quad (36a)$$

that is, the difference is of higher order than \mathbf{u}_1^* and \mathbf{u}^+ both of which are $O(\rho')$. Since Eq.(33) and (35f) are also correct up to $O(\rho')$, in (29) we can substitute

$$\mathbf{u}'' \rightarrow \mathbf{u}' \quad (36b)$$

which implies that in (33a) we can substitute

$$\mathbf{u}_1 = \mathbf{u}^+ \quad (36c)$$

Thus, the density bolus velocity in level coordinates \mathbf{u}^+ coincides with \mathbf{u}_1 defined by (29).

VI. Tracer subgrid in Level Coordinates

Eq.(33) contains two functions that need modeling, a velocity \mathbf{u}_1 and a diffusion-like term S_0 . In paper IV, we modeled \mathbf{u}_1 and its form is given by Eq.(26). Before substituting it into (33), we must transform it to level coordinates. Using Eq.(32), it becomes

$$\mathbf{u}_1 = \kappa \frac{\partial}{\partial z} (\bar{\rho}_z^{-1} \nabla \rho) + \kappa f^{-1} \nabla f \quad (37a)$$

where we have taken into account that in the case of coarse resolution O-GCMs, we can neglect ζ compared to f , as well as the fact that $f_z = 0$. To derive S_0 , we first recall that the general result (33) is valid to second order in the fluctuating fields and to first order in ρ' . Within this accuracy, we can use Eq.(27) of IV which we compare with Eq.(28a) to conclude that:

$$S_0 = \hat{R} \quad (37b)$$

where, within the same approximation, \hat{R} which is given by Eqs.(8a) and (28) of IV, acquires the following form

$$\hat{R} = -\bar{\rho}_z \nabla_\rho \cdot (\kappa \bar{z}_\rho \nabla_\rho \hat{\tau}) \quad (37c)$$

where $\hat{\tau}$ is the thickness weighted average in isopycnal coordinates, see Eq.(4a) of IV, which consistently with the approximation we have used, can be substituted with $\bar{\tau}$ that is,

$$\hat{R} = R = -\bar{\rho}_z \nabla_\rho \cdot (\kappa \bar{z}_\rho \nabla_\rho \bar{\tau}) \quad (37d)$$

Transforming this expression to level coordinates using Eq.(32), we obtain

$$S_0 = R = -\frac{\partial}{\partial x_i} (K_{ij}^0) \frac{\partial \bar{\tau}}{\partial x_j} \quad (37e)$$

where $(\rho_i \equiv \partial \rho / \partial x_i)$

$$\kappa^{-1} K_{ij}^0 = \delta_{ij} - \rho_i \rho_j \rho_k^{-2} \quad (37f)$$

Thus, substituting Eqs.(37a,d,e) into (33a), we can present result in two alternatives forms:

$(\bar{\rho} \equiv \rho, \bar{\tau} \equiv \tau)$:

1) *first form*:

$$S_\ell(\tau) = w_1 \tau_z + (\tau_z / \rho_z) \mathbf{u}_1 \cdot \nabla \rho + S_* + R \quad (38a)$$

$$\mathbf{u}_1 = \kappa \partial_z (\rho_z^{-1} \nabla \rho) + \kappa f^{-1} \nabla f, \quad w_1 = \nabla \cdot \mathbf{I} \quad (38b)$$

$$S_* = \mathbf{I} \cdot \partial_z [\tau_z (\mathbf{L}_\rho - \mathbf{L}_\tau)] \quad (38c)$$

$$\mathbf{L}_\rho = -\rho_z^{-1} \nabla \rho, \quad \mathbf{L}_\tau = -\tau_z^{-1} \nabla \tau \quad (38d)$$

$$\mathbf{I} = -\int_{-H}^z \mathbf{u}_1(z') dz' \quad (38e)$$

$$R = -\frac{\partial}{\partial x_i} (K_{ij} \frac{\partial \tau}{\partial x_j}) \quad (38f)$$

$$\kappa^{-1}K_{ij} = \delta_{ij} - (1-\epsilon)\rho_i\rho_j\rho_k^{-2} \quad (38g)$$

where we have added the diapycnal component with diffusivity $\kappa\epsilon$ which is the vertical diffusivity. The first two terms in (38a) cannot be interpreted as an advection, as in the Gent and McWilliams model (1990, GM). However, we can rearrange (33a) to exhibit such term. To this end, we rewrite (37d) as follows:

$$R = R_* - \kappa\rho_z(\nabla_\rho\rho_z^{-1}) \cdot \nabla_\rho\tau = R_* + \mathbf{u}_1 \cdot \nabla_\rho\tau - \kappa f^{-1}\nabla f \cdot \nabla_\rho\tau \quad (39a)$$

where

$$R_* = -\nabla_\rho \cdot (\kappa\nabla_\rho\tau) - \frac{\partial}{\partial x_i}(\epsilon\kappa\rho_i\rho_j\rho_k^{-2}\frac{\partial\tau}{\partial x_j}) \quad (39b)$$

Using (21a), we obtain:

2) *second form*: $\mathbf{v}_1 = (\mathbf{u}_1, \mathbf{w}_1)$

$$S_\ell(\tau) = \mathbf{v}_1 \cdot \nabla_3\tau + S_* + R_* - \kappa f^{-1}\nabla f \cdot (\nabla + \mathbf{L}_\rho\partial_z)\tau \quad (40a)$$

$$R_* = -(\nabla + \mathbf{L}_\rho\partial_z) \cdot \kappa(\nabla + \mathbf{L}_\rho\partial_z)\tau - \frac{\partial}{\partial x_i}(\epsilon\kappa\rho_i\rho_j\rho_k^{-2}\frac{\partial\tau}{\partial x_j}) \quad (40b)$$

The expressions for \mathbf{u}_1 , S_* , \mathbf{w}_1 and \mathbf{I} are the same as in Eqs.(38).

Since the first term in (40a) looks like an advection term whereas the first term in (38a) does not, two questions arise: first, is a division of the level subgrid in advective plus diffusive parts unique representing a physical reality or is just a formal mathematical reformulation?, second, if the answer is positive, is the first term in (40a) the physical advection term? We discuss this problem in the next section

3) *the GM model*

If we take a constant diffusivity κ as GM did, we obtain from (38)

$$S_\ell(\tau) = S_\ell(\text{GM}, \tau) + \kappa \frac{\partial}{\partial z} [\tau_z \mathbf{L}_\rho \cdot (\mathbf{L}_\rho - \mathbf{L}_\tau)] \quad (40c)$$

which still does not coincide with the GM model. Only if we further take $\tau=\rho$, will the last term vanish and the two models coincide.

VII. Diffusion and Advection in Subgrid Modeling

To discuss the problems just mentioned, we recall that following Taylor's (1915)

ideas, Rhines and Holland (1979, Eqs.4–9a) have shown that equation (19a) ($\bar{\tau} \equiv \tau$)

$$\frac{D\tau}{Dt} = -\frac{\partial}{\partial x_i}(\bar{v}_i \bar{\tau}) = -S_\ell \quad (41a)$$

can be written quite generally as:

$$\bar{v}_i \bar{\tau} = -D_{ij} \frac{\partial \tau}{\partial x_j}, \quad S_\ell = -\frac{\partial}{\partial x_i}(D_{ij} \frac{\partial \tau}{\partial x_j}) \quad (41b)$$

where the diffusivity D_{ij} is a second-order tensor the orientation of which is unknown. As stressed by Muller and Holloway (1989), Greatbatch (1998), the principal axes may be taken to be diagonal in level, isopycnals or any other coordinate systems. For a number of years, the tensor was assumed to be diagonal in the (xyz) space giving rise to horizontal/vertical diffusivities. Redi (1982) suggested that it is diagonal in the isopycnal/diapycnal system and since mesoscale mixing occur along isopycnals, the tensor has ever since been taken diagonal in that system and thus one refers to diapycnal and isopycnal diffusivities. Suppose that D_{ij} is a symmetric tensor and that we take it equal to Redi's K_{ij} , Eq.(38g):

$$D_{ij} = K_{ij} \quad (41c)$$

where

$$K_{ij} = \kappa(\delta_{ij} - \rho_i \rho_j \rho_k^{-2} + \epsilon \rho_i \rho_j \rho_k^{-2}) = K_{ij}^0 + \epsilon K_{ij}(D) \quad (41d)$$

The first two terms represent the isopycnal contribution since $K_{ij}^0 \rho_j = 0$, while the last term represents the diapycnal contribution since $K_{ij}(D) \rho_j = \rho_i$. The question is: can the velocity

$$v_i^{++} = \frac{\partial}{\partial x_j} K_{ij} \quad (41e)$$

be interpreted as the bolus velocity $v_i^{++}(\mathbf{u}_i^{++}, \mathbf{w}_i^{++})$, see Eq.(1b) of paper IV? The horizontal and vertical components are:

$$-\frac{\partial}{\partial z}(\kappa \rho_z^{-1} \nabla \rho) + \dots, \quad -\nabla \cdot (\kappa \rho_z^{-1} \nabla \rho) \quad (41f)$$

which have the same functional dependence of the GM model bolus velocity but with the difference that both velocities (41f) are either from low to high ϖ or vice-versa, whereas in the GM model the horizontal part is from low to high ϖ while the vertical is from high to low ϖ . In more mathematical terms, the true bolus velocity is divergence free (by

construction) while (41f) is not. Next, we write D_{ij} as the sum of a symmetric part, \mathfrak{P}_{ij} and an antisymmetric part a_{ij} (Plumb, 1979):

$$D_{ij} = \kappa_{ij} + a_{ij} \quad (42a)$$

Thus,

$$\overline{v_i^+ \tau} = -\kappa_{ij} \frac{\partial \tau}{\partial x_j} + \tau \frac{\partial}{\partial x_j} a_{ij} - \frac{\partial}{\partial x_j} (\tau a_{ij}) \quad (42b)$$

Contrary to (41e), we now consider $\partial/\partial x_j$ of a_{ij} as a possible candidate for the advective velocity v_i^{++}

$$v_i^{++} = \frac{\partial}{\partial x_j} a_{ij} \quad (42c)$$

Since $a_{ij} = -a_{ji}$, it follows that the last term in (42b) is such that

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\tau a_{ij}) = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\tau a_{ji}) = 0 \quad (42d)$$

and thus it does not contribute to S_ℓ in (41a). By the same token, it follows that v_i^{++} is divergence free

$$\frac{\partial}{\partial x_i} v_i^{++} = 0 \quad (42e)$$

and so going back to (41b) we have

$$S_\ell = v_i^{++} \frac{\partial \tau}{\partial x_i} - \frac{\partial}{\partial x_i} (\kappa_{ij} \frac{\partial \tau}{\partial x_j}) \quad (42f)$$

This shows that the symmetric part is a diffusion process while the antisymmetric part gives rise to an advection velocity. Thus, if D_{ij} is known somehow, the decomposition of S_ℓ into advective and diffusive terms is unique since for any tensor the decomposition in symmetric and antisymmetric parts is unique. If, on the contrary, D_{ij} is not known but S_ℓ is, as in our case where we have constructed Eqs.(38) and (40), the uniqueness of the decomposition of S_ℓ is predicated on D_{ij} depending only on ρ but not on τ . First, let us use (42a) into (41b). We obtain (where commas indicated derivatives):

$$S_\ell = -\kappa_{ij} \tau_{,ij} - (\kappa_{ij,i} + a_{ij,i}) \tau_{,j} \quad (42g)$$

By equating the terms in the known S_ℓ which contain the second derivatives of τ with the first term in (42g), one can construct the tensor κ_{ij} . To obtain the last two term in (42f),

we equate them with the corresponding terms in S_ℓ which contain the first derivative of τ . This will yield $a_{ij,i}$. In summary, by knowing S_ℓ we can uniquely reconstruct from it both the tensor κ_{ij} as well as $a_{ij,i}$ which is nothing but $-v_j^{++}$, as Eq.(42c) shows. Applying this method using the specific form (38) and/or (40), we obtain the following results:

$$S_\ell(\tau) = S_{\text{diff}}(\tau) + S_{\text{adv}}(\tau) \quad (43a)$$

$$S_{\text{diff}}(\tau) = -\frac{\partial}{\partial x_i}(\kappa_{ij} \frac{\partial \tau}{\partial x_j}) \quad (43b)$$

$$S_{\text{adv}}(\tau) = v_i^{++} \frac{\partial \tau}{\partial x_i} \quad (43c)$$

$$v_i^{++} = \frac{\partial}{\partial x_j} a_{ij} \quad (43d)$$

where:

$$\mathbf{u}_i^{++} = \frac{1}{2} \mathbf{u}_1 \quad (44a)$$

$$\nabla \mathbf{u}_i^{++} + \mathbf{w}^{++} = 0 \quad (44b)$$

$$\kappa_{ij} = K_{ij} + \Delta_{ij} \quad (44c)$$

$$\Delta_{\alpha\beta} = 0, \quad (\alpha, \beta = 1, 2) \quad (44d)$$

$$\Delta_{\alpha 3} = \Delta_{3\alpha} = -\frac{1}{2} \mathbf{I}_\alpha, \quad (\alpha = 1, 2) \quad (44e)$$

$$\Delta_{33} = \rho_z^{-1} \mathbf{I} \cdot \nabla \rho \quad (44f)$$

Thus, we reach the following conclusions: *first*, separation (43a) in terms of advective and diffusive components is unique; *second*, the velocity \mathbf{v}_1 in the first term in (40a) is not the physically correct bolus velocity since it is twice as large as the real (advective) bolus velocity $(\mathbf{u}_i^{++}, \mathbf{w}_i^{++})$, Eqs.(43a)–(44a); *third*, the diffusion tensor κ_{ij} is not only the Redi K_{ij} , as assumed thus far, since it contains the additional term Δ_{ij} . Since the velocity \mathbf{u}_1 equal the density bolus velocity, Eq.(36c), it may at first seem surprising that in level coordinates

$$\mathbf{u}_i^{++}(\text{tracer}) = \frac{1}{2} \mathbf{u}^+(\text{thickness}) \quad (44g)$$

instead of

$$\mathbf{u}_i^{++}(\text{tracer}) = \mathbf{u}^+(\text{thickness}) \quad (44h)$$

We now show that (44g) and (44h) represent two different interpretations of the physical

content of the density subgrid. If we substitute $\tau \rightarrow \rho$ in (43) we obtain

$$\begin{aligned} S_{\rho}(\rho) &= S_{\text{diff}}(\rho) + S_{\text{adv}}(\rho) \\ S_{\text{diff}}(\rho) &= -\frac{\partial}{\partial x_i}(\kappa_{ij} \frac{\partial \rho}{\partial x_j}), \quad S_{\text{adv}}(\rho) = v_i^{++} \frac{\partial \rho}{\partial x_i} \end{aligned} \quad (44i)$$

Using now (43d) and (44a–f), we obtain:

$$S_{\text{diff}}(\rho) = S_{\text{adv}}(\rho) = \frac{1}{2}(\mathbf{u}_1 \cdot \nabla \rho + w_1 \rho_z) \quad (44j)$$

$$S_{\rho}(\rho) = \mathbf{u}_1 \cdot \nabla \rho + w_1 \rho_z \quad (44k)$$

Eq.(44j) can be interpreted in only one way: density subgrid is the sum of advection plus diffusion which contribute an equal amount. On the other hand, (44k) can be interpreted in two ways: either as the sum of two equal terms or, as the absence of diffusion and full contribution of advection. *The first choice implies (44h) while the second choice implies (44g).* Thus, we conclude that while both (44g) and (44h) are acceptable, the common interpretation is that for density there is no diffusion and thus (44g) is to be preferred. Stated differently, the standard model that:

$$S_{\text{diff}}(\rho)=0, \quad \mathbf{u}_i^{++}=\mathbf{u}^+$$

is untenable since the correct relations are either

$$S_{\text{diff}}(\rho)=0, \quad \mathbf{u}_i^{++} = \frac{1}{2}\mathbf{u}^+$$

or:

$$S_{\text{diff}}(\rho) = S_{\text{adv}}(\rho) \neq 0, \quad \mathbf{u}_i^{++}=\mathbf{u}^+$$

The question still remains as to why for density, the separation of advective from diffusive terms is not unique. The answer is that in the proof of uniqueness given above, the critical condition was that D_{ij} be independent of τ which is no longer true when $\tau \rightarrow \rho$ since Eq.(44c–f) show that D_{ij} via κ_{ij} depends on ρ .

VIII. Present and previous models

In principle, there are four bolus velocities: for density and tracer and in level and isopycnal coordinates, that is,

$$\text{density: } \mathbf{u}^+(\text{level}), \quad \mathbf{u}^*(\text{iso}) \quad (45a)$$

$$\text{tracer: } \mathbf{u}_i^{++}(\text{level}), \quad \mathbf{u}^{**}(\text{iso}) \quad (45b)$$

In terms of these velocities, the subgrid tracer functions $S_{\ell,1}$ are defined as:

$$S_\ell = \mathbf{u}_i^{++} \cdot \nabla \bar{\tau} + \mathbf{w}_i^{++-} \tau_z + R(\text{level}) \quad (46a)$$

$$S_1 = \mathbf{u}^{**} \cdot \nabla_\rho \tilde{\tau} + R(\text{iso}) \quad (46b)$$

In all previous models it was assumed that all four bolus velocities are identical:

$$\mathbf{u}^{**}(\text{iso}) = \mathbf{u}^*(\text{iso}) = \mathbf{u}^+(\text{level}) = \mathbf{u}_i^{++}(\text{level}) \quad (46c)$$

whereas we have shown that:

$$\mathbf{u}^*(\text{iso}) = \mathbf{u}^{**}(\text{iso}) \quad (47a)$$

$$\mathbf{u}_i^{++}(\text{level}) = \frac{1}{2} \mathbf{u}^+(\text{level}) \quad (47b)$$

$$\mathbf{u}^*(\text{iso}) = \mathbf{u}^+(\text{level}) + \mathbf{u}_2 \quad (47c)$$

where \mathbf{u}_2 is given by Eq.(24) of IV. To understand how the second equality in (46c) was arrived at, we begin with the equation for the thickness $z_\rho = \partial z / \partial \rho$ whose mean value in isopycnal coordinates satisfies the equation:

$$\partial_t \tilde{z}_\rho + \nabla_\rho \cdot [(\tilde{\mathbf{u}} + \mathbf{u}^*) \tilde{z}_\rho] = 0 \quad (48a)$$

where

$$\mathbf{u}^* \equiv \overline{\mathbf{u}'' z'_\rho} / \tilde{z}_\rho \quad (48b)$$

is the *thickness bolus velocity in isopycnal coordinates*. Gent and al. (1995) transformed (48a) to level coordinates to obtain

$$\partial_t \rho + (\tilde{\mathbf{u}} + \mathbf{u}^*) \cdot \nabla \rho + (\tilde{w} + \mathbf{w}^*) \partial_z \rho = 0 \quad (49a)$$

Within the $O(\rho')$ accuracy, in (49a) one takes $\rho = \bar{\rho}$, a variable for which we have already given the transport equation (35f) which we reproduce here for ease of comparison:

$$\partial_t \bar{\rho} + (\bar{\mathbf{u}} + \mathbf{u}^+) \cdot \nabla \bar{\rho} + (\bar{w} + \mathbf{w}^+) \bar{\rho}_z = 0 \quad (49b)$$

By comparing (49a,b), one can only conclude that

$$\begin{aligned} \tilde{\mathbf{u}} + \mathbf{u}^* &= \bar{\mathbf{u}} + \mathbf{u}^+ \\ \tilde{w} + \mathbf{w}^* &= \bar{w} + \mathbf{w}^+ \end{aligned} \quad (49c)$$

whereas in the GM model

$$\mathbf{u}^* = \mathbf{u}^+ \quad (49d)$$

which is the assumption we referred to. Thus, we see that (49d) is due to the lack of differentiation between $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ [in our model, substituting Eqs.(35b), (48b) and (11a) into (18a), one obtains the first of (49c)].

To understand whence the last assumption (46c) in the GM model arises, we recall that GM heuristically extended (49a) to the case of a tracer by adding a diffusion term R thus obtaining (see also Gent et al., 1995)

$$\partial_t \bar{\tau} + \bar{\mathbf{u}} \cdot \nabla \bar{\tau} + \bar{w} \bar{\tau}_z + S_\ell = 0 \quad (50a)$$

$$S_\ell(\text{GM}) = \mathbf{u}^* \cdot \nabla \bar{\tau} + w^* \bar{\tau}_z + R \quad (50b)$$

Eq.(50b) is consistent with (46c) only if ones makes the assumption

$$\mathbf{u}_i^{++} = \mathbf{u}^* \quad (50c)$$

IX. Conclusions.

The goal of this paper was to improve on the "effective $\rho(z)$ approximation", employed thus far in all considerations, since it neglects the intrinsic random nature of the function $\rho(z)$. We began by considering the transformation of the mean and fluctuating parts of the different fields of interest from isopycnal to level coordinates. For the velocity field, the results are presented in Eqs.(16) and (18) which are accurate up to the second-order in the fluctuating fields. Analogous transformations for the tracer fields can be obtained via the substitution $\mathbf{v} \rightarrow \tau$. Using these results, we compute the subgrid term S_ℓ that appears in Equation (19a) for the mean tracer $\bar{\tau}$ in level coordinates. *The main results are expressed by Eqs.(38) and (40) and/or (43)–(44).* A considerable simplification of the problem was achieved thanks to the experimental results by Gargett et al. (1981) that showed that the vertical correlation length scale for ocean turbulence (below the mixed layer) is much smaller than the characteristic vertical extent, H . As we have shown, this feature leads to Eqs.(24,25) which in turn imply that ocean mixing in stably stratified

regions occurs mainly along isopycnals. On this basis we derived our main results which are quite different from the GM model, Eq.(50b).

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